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Information Properties of Generalized Order Statistics and Renyie Information

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Abstract

Our aim at this paper is to investigate properties of Shannon and REnyie entropy, Kullback-Leibler (K-L) information and mutual information of generalized order statistics (GOS). We show that discrimination information and Renyie information between distribution of GOS and parent distribution, the discrimination information among the GOS and the mutual information between GOS are all distribution free. We also discuss Renyie information properties of GOS. Some bounds for K-L information is constructed.

Keywords: Generalized Order Statistics; differential entropy; Renyie Information; Kullback-Leibler information; mutual information.

Introduction and notation

Kamps (1995) introduced generalized order statistics as a random variable having special properties and certain joint density functions of ordered random variables such as order statistics, k-records and etc. GOS provide unified approach to a variety of models of ordered random variables with different interpretations, such as ordinary order statistics, sequential order statistics, progressively type II censored order statistics, record values, k_{th} record values, and Pfeifer`s records.

Order statistics and records have been used in a wide range of problems, including statistical estimation and prediction, characterization of probability distributions, seismology, detection of outliers and goodness of fit tests. They can be consider as special cases of GOS. See for example Arnold et al. (1992, 1998), David and Nagaraja (2003).

Information properties of records and order statistics have investigated by Baratpur et al. (2007) and Ebrahimi et al. (2004), also Renyie (see Renyie (1961)) information properties explored by some authors (see Abbasnejad and Arghami (2011)). Information properties of GOS are investigated at this paper.

Suppose that $X_{r;n,m,k}$ be the r_{th} GOS, so if $X_{1;n,m,k}, X_{2;n,m,k}, ..., X_{n;n,m,k}$ be n GOS from the cdf F(x), where $n > 1$, $m \geq -1$, $k \geq 1$ are real numbers. Joint pdf of

$f_{X_{1;n,m,k}, X_{2;n,m,k}, ..., X_{n;n,m,k}}(x_1, x_2, ..., x_n)$ is given by

$$f_{X_{1;n,m,k}, X_{2;n,m,k}, ..., X_{n;n,m,k}}(x_1, x_2, ..., x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^m f(x_i)\right)(1-F(x_n))^{k-1} f(x_n),$$

where $\gamma_j = k + (n-j)(m+1),\ j = 1, 2, ..., n.$ Also pdf of $f_{X_{r;n,m,k}}(x)$ is given by

$$f_{X_{r;n,m,k}}(x) = \frac{\prod_{j=1}^{r}\gamma_j}{(r-1)!(m+1)^{r-1}}(\overline{F}(x))^{\gamma_r-1} f(x)(1-\overline{F}^{m+1}(x))^{r-1}, \tag{1}$$

We define $A_{r;n,m,k} = \dfrac{\prod_{j=1}^{r}\gamma_j}{(r-1)!(m+1)^{r-1}}$, for simplicity, we shall write $A_{r;n,m,k} = A_r$,

Also we have

$$g_m(x) = h_m(x) - h_m(0) = \begin{cases} \frac{1}{m+1}(1-(1-x)^{m+1}), & m \neq -1 \\ -ln(1-x) & m=-1, x\in[0,1) \end{cases}$$

where

$$h_m(x) = \begin{cases} \frac{-1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -ln(1-x) & m=-1, x\in[0,1) \end{cases}.$$

The joint *pdf* of $X_{r;n,m,k}, X_{s;n,m,k}, 1 \le r < s \le n$ is given by

$$f_{r,s;n,m,k}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}(\overline{F}(x))^m f(x) g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\overline{F}(y))^{\gamma_s-1} f(x), \qquad x < y. \tag{2}$$

At this paper we explore informational properties of GOS, including *Shannon* and Renyie entropy, *Kullback – Leibler* information, mutual and Renyie Information. Also we show that some of this properties are generalizations of results contained in Ebrahimi et al. (2004) and Park (1995). It is well known that *Shannon* and Renyie entropy and K-L, mutual and Renyie information are defined as follows, respectively:

$$H(X) = -\int_{-\infty}^{+\infty} f(x) log f(x) dx, \tag{3}$$

$$H_\alpha(X) = \frac{-1}{\alpha-1}\int_{-\infty}^{+\infty} f^\alpha(x) dx, \tag{4}$$

$$D_n(f_i(x) : f_j(x)) = \int_{-\infty}^{+\infty} f_i(x) log \frac{f_i(x)}{f_j(x)} dx, \tag{5}$$

$$M_n(X,Y) = D_n(f(x,y) : f(x)f(y)), \tag{6}$$

$$D_n^\alpha(f(x) : g(x)) = \frac{1}{(\alpha-1)} log \int_{-\infty}^{+\infty} \{\frac{f(x)}{g(x)}\}^{\alpha-1} f(x) dx. \tag{7}$$

Hereafter, the range of integration will not be shown and should be clear from the context.

The rest of the paper is organized as follows. In the section (II) some results on the entropy of GOS are presented. Section (III) presents some results on the discrimination information function based on GOS. Section (IV) gives mutual information properties of GOS. In the last section some results for Renyie entropy and information of GOS are obtained.

Entropy Of Generalized Order Statistics

At this section we explore the entropy of GOS. Using (1) and (3) we have

$$H_r(X_r) = -\int f_{X_{r;n,m,k}}(x) log f_{X_{r;n,m,k}}(x) dx,$$

using the fact that $f_{X_{r;n,m,k}}(x)$ is a density function so,

$$\int f_{X_{r;n,m,k}}(x) dx = 1$$

Also it can be derived that

$$\int f_{X_{r;n,m,k}}(x) log \overline{F}(x) dx = \frac{1}{m+1}(\psi(\frac{\gamma_r}{m+1}) - \psi(r + \frac{\gamma_r}{m+1})),$$

where $\psi(w) = \frac{d\Gamma(w)}{dw}$, is the digamma function and $\Gamma(w) = \int_0^{+\infty} x^{w-1} e^{-x} dx$, is gamma function. Also beta function defined by $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, and after some calculation it can be obtained that,

$$\frac{A_r B(r, \frac{\gamma_r}{m+1})}{m+1} = 1, \tag{8}$$

so we can obtain

$$\int f_{X_{r;n,m,k}}(x) log(1 - \overline{F}^{m+1}(x)) dx = \psi(r) - \psi(r + \frac{\gamma_r}{m+1}).$$

Using results above, we get to

$$H_n(X_r) = -\{log A_r + \frac{\gamma_r - 1}{m+1}(\psi(\frac{\gamma_r}{m+1}) - \psi(r + \frac{\gamma_r}{m+1})) + (r-1)(\psi(r) - \psi(r + \frac{\gamma_r}{m+1}))\}$$
$$- \int f_{X_{r;n,m,k}}(x) log f(x) dx. \tag{9}$$

Entropy of first and last GOS is obtained as follows:

$$H_n(X_1) = 1 - log\gamma_1 - 1/\gamma_1 - \int f_{X_{1;n,m,k}}(x) log f(x) dx$$

$$H_n(X_n) = -\{log A_n + \frac{k-1}{m+1}(\psi(\frac{k}{m+1}) - \psi(n + \frac{k}{m+1})) + (n-1)(\psi(n) - \psi(n + \frac{k}{m+1}))\}$$
$$- \int f_{X_{n;n,m,k}}(x) log f(x) dx.$$

If we set $k = 1$, and $m = 0$ results for ordinary order statistics (o'OS) could be obtained, so this results are generalizations of results contained in Ebrahimi et al. (2004) and Park (1995). Also if we take $k = R + 1$, and $m = 0$ results for entropy of censored data would be extracted. If U has standard uniform distribution with density $g(u) = 1, u \in [0,1]$ then

$$H_n(U_r) = -\{log A_r + \frac{\gamma_r - 1}{m+1}(\psi(\frac{\gamma_r}{m+1}) - \psi(r + \frac{\gamma_r}{m+1})) + (r-1)(\psi(r) - \psi(r + \frac{\gamma_r}{m+1}))\}, \tag{10}$$

so

$$H_n(X_r) = H_n(U_r) - \int f_{X_{r;n,m,k}}(x) log f(x) dx$$

The following property of the standard uniform distribution is used in the sequel. Let

$$\Delta_n(r) = H_n(U_{r+1}) - H_n(U_r) = [log\, r - \psi(r)] - [log \frac{\gamma_{r+1}}{m+1} - \psi(\frac{\gamma_{r+1}}{m+1})] - \frac{m}{\gamma_{r+1}}. \tag{11}$$

The equality is obtained by noting that $r + 1 + \frac{\gamma_{r+1}}{m+1} = r + \frac{\gamma_r}{m+1}$, $\psi(n) - \psi(n-1) = \frac{1}{n-1}$, $\gamma_r - 1 - \gamma_{r+1} = m$, and $\frac{A_r}{A_{r+1}} = \frac{r(m+1)}{\gamma_{r+1}}$. We could obtain

$$\Delta_n(r) + \frac{m}{\gamma_{r+1}} < 0, \qquad for \qquad r < \frac{1}{2}(\frac{\gamma_1}{m+1})$$

$$\Delta_n(r)+\frac{m}{\gamma_{r+1}}>0, \qquad for \qquad r>\frac{1}{2}(\frac{\gamma_1}{m+1}) \tag{12}$$

The inequalities (12) are obtained by noting that $\frac{1}{w}-\psi'(w)<0$ where ψ' is the trigamma function, see sandor (2005), pp 198-199, so $log\,w-\psi(w)$ is a decreasing function. Readily, we can show that $\Delta_n(\frac{1}{2}(\frac{\gamma_1}{m+1}))=0.$ An even $n,$ $k=1,$ and $m=0,$ resulted that $r=\frac{n}{2}$ which is median.

Theorem 2.1 For any non-negative random variable X, with cdf $F(x)$, and real numbers m, k with $m\geq -1$, $k\geq 1$ and integer $r\geq 1$, the relation

$$H_n(X_{r+1})-H_n(X_r)=\Delta_n(r)+\int f_{X_{r;n,m,k}}(x)logf(x)dx-\int f_{X_{r+1;n,m,k}}(x)logf(x)dx$$

is satisfied.

Proof.

The proof could be obtained using relations (9) and (10), and it is ommited.

Example 2.2 Let X be a random variable having the exponential distribution $F(x)=1-e^{-\lambda x}$. we write

$$\int f_{X_{r;n,m,k}}(x)logf(x)dx=log\lambda+[\psi(n-r+1)-\psi(n+1)]$$

then

$$H_n(X_{r+1})-H_n(X_r)=\frac{1}{n-r}-\Delta_n(r)\geq 0$$

The inequality can be seen at section (III). So, the entropy of r_{th} GOS of exponential distribution is increasing in r.

Discrimination Information

At this section discrimination information between distribution of GOS and parent distribution and the discrimination information between the distribution of the GOS are discussed.

For simplicity we define $f_{X_{r;n,m,k}}(x)=f_r(x),$ also from (5) we calculate K-L information between distribution of GOS and the data distribution so we have

$$D_n(f_r(x):f(x))=\int f_r(x)logf_r(x)f(x)dx.$$

Using relations (9) and (10) we have

$$D_n(f_r(x):f(x))=-H_n(U_r)$$

where U_r has standard uniform distribution. Therefore, according to relation (10) the discrimination information between the distribution of GOS and the parent distribution is distribution free. Also Using relation (11), it can be seen that

$$D_n(f_{r+1}(x):f(x))-D_n(f_r(x):f(x))=\Delta_n(r).$$

Using (12), it concludes that among the GOS $[\frac{1}{2}(n-1+\frac{k}{m+1})]_{th}$ GOS has the closest distribution to the data distribution. Also, as previously noted, in the case of o'OS, for an even $n,$ $\Delta_n(\frac{n}{2})=0,$ so among the order statistics the median has the closest distribution to the parent distribution.

The K-L information between r_{th} and s_{th} GOS is given by

$$D_n(f_r(x): f_s(x)) = \int f_r(x) log f_r(x) f_s(x) dx.$$

According to relation (1) we could write

$$\frac{f_r(x)}{f_s(x)} = \frac{A_r}{A_s}[\overline{F}(x)]^{\gamma_r - \gamma_s}(1-\overline{F}^{m+1}(x))^{r-s}$$

which, $\gamma_r - \gamma_s = (s-r)(m+1)$, and using relation (8) we have $\frac{A_r}{A_s} = \frac{B(s,\frac{\gamma_s}{m+1})}{B(r,\frac{\gamma_r}{m+1})}$ So, it shall be written

$$D_n(f_r(x): f_s(x)) = log\frac{B(s,\frac{\gamma_s}{m+1})}{B(r,\frac{\gamma_r}{m+1})} + (r-s)[\psi(r) - \psi(\frac{\gamma_r}{m+1})]. \qquad (13)$$

According to relation (13) it can be derived K-L information for consecutive GOS as

$$D_n(f_r(x): f_{r+1}(x)) = log\frac{r(m+1)}{\gamma_{r+1}} - [\psi(r) - \psi(\frac{\gamma_r}{m+1})] = \Delta_n(r) + \frac{m}{\gamma_{r+1}}. \qquad (14)$$

last equality comes from the relation (11). Also, it can be written

$$D_n(f_{r+1}(x): f_r(x)) = log\frac{\gamma_{r+1}}{r(m+1)} + [\psi(r+1) - \psi(\frac{\gamma_{r+1}}{m+1})] = -[\Delta_n(r) + \frac{m}{\gamma_{r+1}}] + \frac{1}{r} + \frac{\gamma_{r+1}}{m+1}, \qquad (15)$$

equality for relation (15) comes from relation (14) and property of digamma function which noted previously.

Now, according to this results the symmetric divergence could be obtained as follows

$$J_n(f_{r+1}(x), f_r(x)) = D_n(f_r(x): f_{r+1}(x)) + D_n(f_{r+1}(x): f_r(x)) = \frac{\gamma_1}{r\gamma_{r+1}}.$$

Next we investigate ordering properties of distributions based on discrimination information of GOS. We need some definitions in which X and Y denote random variables with distribution functions F_X and F_Y.

Definition 3.1 The random variable X is said to be stochastically less than or equal to the random variable Y, denoted by $X \leq_{st} Y$ if $\overline{F}_X(z) \leq \overline{F}_Y(z)$, for all z.

Definition 3.2 The random variable X is said to be smaller in the likelihood ratio ordering than the random variable Y, denoted by $X \leq_{lr} Y$, if and only if there are densities f and g of corresponding random variables such that $f(u)g(v) \geq f(v)g(u)$, for all $u \leq v$, which means $f_X(x)/f_Y(x)$ is nondecreasing in x.

it is well known that $X \leq_{lr} Y$ implies $X \leq_{st} Y$.

Theorem 3.3 For $m > 0$, let X and Y be two random variables and let W_r and Z_r, $r = 1,\cdots,n$ be their GOS with densities f_r and g_r, respectively.

a) if $Y \leq_{st} W_r$ and $X \leq_{lr} Z_r$ then $D_n(f_{r+1}: g_{r+1}) \leq D_n(f_r: g_r)$ for $r \leq \frac{1}{2}\frac{\gamma_1}{m+1}$.

b) if $Y \geq_{st} W_{r+1}$ and $X \geq_{lr} Z_{r+1}$ then $D_n(f_{r+1}: g_{r+1}) \geq D_n(f_r: g_r)$ for $r \geq \frac{1}{2}\frac{\gamma_1}{m+1}$.

Proof.

a) write

$$D_n(f_r : g_r) = \int f_r(x) log \frac{f_r(x)}{g_r(x)} dx = \int f_r(x) log \frac{f_r(x)}{f(x)} \frac{f(x)}{f_r(x)} dx =$$

$$\int f_r(x) log \frac{f_r(x)}{f(x)} dx + \int f_r(x) log \frac{f(x)}{g_r(x)} dx$$

$$= D_n(f_r : f) + \int f_r(x) log \frac{f(x)}{g_r(x)} dx$$

Therefore,

$$D_n(f_{r+1} : g_{r+1}) - D_n(f_r : g_r) = D_n(f_{r+1} : f) + \int f_{r+1}(x) log \frac{f(x)}{g_{r+1}(x)} dx - D_n(f_r : f)$$

$$- \int f_r(x) log \frac{f(x)}{g_r(x)} dx$$

$$= \Delta_n(r) + \int f_{r+1}(x) log \frac{f(x)}{g_{r+1}(x)} dx - \int f_r(x) log \frac{f(x)}{g_r(x)} dx$$

$$\le -\frac{m}{\gamma_{r+1}} + \int f_{r+1}(x) log \frac{f(x)}{g_{r+1}(x)} dx - \int f_r(x) log \frac{f(x)}{g_r(x)} dx$$

$$\le \int f_{r+1}(x) log \frac{f(x)}{g_{r+1}(x)} dx - \int f_r(x) log \frac{f(x)}{g_r(x)} dx \le \int f_{r+1}(x) log \frac{f(x)}{g_{r+1}(x)} dx - \int f_{r+1}(x) log \frac{f(x)}{g_r(x)} dx \le 0.$$

The first inequality comes from the relation (12), the second inequality is due to the fact that m is positive integer and for any r, $\gamma_r > 0$. The third inequality comes from the fact that $W_r \le_{st} W_{r+1}$. The last inequality follows the fact that $X \le_{lr} Z_r$.

b) The proof is similar to the part a and is omitted.

Mutual Information

At this section, properties of mutual information are explored and we show that this measure of information between consecutive GOS is distribution free. Using relation (2) joint density of r_{th} and $r+1_{th}$ GOS, is

$$f_{r,r+1}(x,y) = \gamma_{r+1} A_r (\overline{F}(x))^m f(x)(1-\overline{F}^{m+1}(x))^{r-1} f(y)(\overline{F}(y))^{\gamma_{r+1}-1} \qquad x < y.$$

Using relation (6) we calculate mutual information between r_{th} and $r+1_{th}$ GOS.

$$M_n(Y_r, Y_{r+1}) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{r,r+1}(x,y) log \frac{f_{r,r+1}(x,y)}{f_r(x) f_{r+1}(y)} dxdy. \tag{16}$$

We show that mutual information between consecutive GOS is distribution free.

Theorem 4.1 Let X be a random variable having distribution $f_X(x)$ and let Y_r $r = 1,\dots,n$,denote its GOS, then the mutual information between consecutive GOS is distribution free and given by

$$M_n(Y_r, Y_{r+1}) = log \frac{r(m+1)}{A_r} + \frac{\gamma_{r+1}^2}{(m+1)^2}[\psi(\frac{\gamma_r}{m+1}) - \psi(r + \frac{\gamma_r}{m+1})]$$

$$- r[\psi(r+1) - \psi(\frac{\gamma_{r+1}}{m+1})] + \frac{\gamma_{r+1}}{m+1}$$

Proof.

Firstly, write

$$\frac{f_{r,r+1}(x,y)}{f_r(x)f_{r+1}(y)}=\frac{r(m+1)}{A_r}\overline{F}^{\gamma_{r+1}}(x)[1-\overline{F}^{m+1}(y)]^{-r}$$

so we have

$$\int_{-\infty}^{y} f(x)\overline{F}^{m}(x)[1-\overline{F}^{m+1}(x)]^{r-1}dx=\frac{[1-\overline{F}^{m+1}(y)]^{r}}{r(m+1)}$$

the equality is obtained using the change of variable $z=1-\overline{F}^{m+1}$. According to the fact that $\frac{d}{dm}\overline{F}^{m}(x)=\overline{F}^{m}(x)log\overline{F}(x),$ we have

$$\int_{-\infty}^{y} f(x)\overline{F}^{m}(x)[1-\overline{F}^{m+1}(x)]^{r-1}log\overline{F}(x)dx$$

$$=-[\frac{1}{m+1}\overline{F}^{m+1}(y)(1-\overline{F}^{m+1}(y))^{r-1}log\overline{F}(y)+\frac{(1-\overline{F}^{m+1}(y))^{r}}{r(m+1)^{2}}].$$

Readily, it can be obtained

$$\int_{-\infty}^{\infty} f(y)\overline{F}^{\gamma_{r+1}-1}(y)(1-\overline{F}^{m+1}(y))^{r}dy=\frac{B(r+1,\frac{\gamma_{r+1}}{m+1})}{m+1},$$

from the fact that $\gamma_{r+1}+m=\gamma_r-1$ we get to

$$\int_{-\infty}^{\infty} f(y)\overline{F}^{\gamma_{r}-1}(y)(1-\overline{F}^{m+1}(y))^{r-1}dy=\frac{B(r,\frac{\gamma_{r}}{m+1})}{(m+1)^{2}}[\psi(\frac{\gamma_r}{m+1})-\psi(r+\frac{\gamma_r}{m+1})],$$

according to the relation $\frac{d}{dr}[1-\overline{F}^{m+1}]^{r}=[1-\overline{F}^{m+1}]^{r}log[1-\overline{F}^{m+1}]$ it can be obtained that

$$\int_{-\infty}^{\infty} f(y)\overline{F}^{\gamma_{r+1}-1}(y)(1-\overline{F}^{m+1}(y))^{r}log(1-\overline{F}^{m+1}(y))dy=\frac{B(r+1,\frac{\gamma_{r+1}}{m+1})}{(m+1)}[\psi(r+1)-\psi(\frac{\gamma_{r+1}}{m+1})],$$

finally, Using the relations $B(r+1,\frac{\gamma_{r+1}}{m+1})=\frac{r(m+1)}{\gamma_{r+1}}B(r,\frac{\gamma_r}{m+1})$ and $A_{r+1}=\frac{\gamma_{r+1}}{r(m+1)}A_r$ and (8), we get to the desired result.

Renyi Information

This section is devoted to investigation of Renyi information properties of GOS. Renyi entropy is a generalization of *Shannon* entropy, see Renyi (1961). The entropy of order α or Renyi entropy of a distribution is defined as relation (4) where $\alpha>0,\alpha\neq 1$. It can be easily shown that $H_\alpha(X)\to H(X)$ as $\alpha\to 1$. Let U be a random variable from the standard uniform distribution, and U_r denoted r_{th} GOS. Then the Renyi entropy of U_r can be expressed as

$$H^{\alpha}_{U_r}=-\frac{1}{\alpha-1}log\frac{A_r^{\alpha}}{m+1}B(\alpha(r-1)+1,\frac{\alpha}{m+1}(\gamma_r-1)-m+1). \quad (17)$$

In the following lemma, we obtain Renyi entropy of r_{th} GOS from an arbitrary distribution in terms of Renyi entropy of r_{th} GOS from standard uniform distribution.

Lemma 5.1 Let X be a random variable from the distribution $F(x)$ and the quantile function $F^{-1}(.)$, and let X_r be r_{th} GOS of random variable X. Then Renyi entropy of X_r can be obtained as

$$H_\alpha(X_r) = H^\alpha_{U_r} - \frac{1}{\alpha-1} log E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z_r)^{\frac{1}{m+1}})]],$$

where Z is a random variable, whose pdf is denoted by $g(z)$ and is distributed as $Beta(\alpha(r-1)+1, \frac{\alpha}{m+1}(\gamma_r-1)-m+1)$.

Proof.

Using formulas (1) and (4), and by transformation $z = 1-\overline{F}^{m+1}(x)$, we have

$$H^\alpha(X_r) = -\frac{1}{\alpha-1} log \frac{A_r^\alpha}{m+1} B(\alpha(r-1)+1, \frac{\alpha}{m+1}(\gamma_r-1)-m+1)$$
$$\times E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})]],$$

by applying relation (17) we get to the desired result.

$f[F^{-1}(.)]$ is called density-quantile function. see David and Nagaraja (2003).

As an application of the lemma (5.1) consider the following example.

Example 5.2 Let X be a random variable with density function $f(x) = \lambda e^{-\lambda x}$. For computing Renyi entropy, we calculate quantile function $F^{-1}(w) = -\lambda log(1-w)$, so density-quantile function term can be expressed as $f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})] = \frac{(1-z^{\frac{\alpha-1}{m+1}})}{\lambda^{\alpha-1}}$.

For simplicity, let $a = \alpha(r-1)+1$ and $b = \frac{\alpha}{m+1}(\gamma_r-1)-m+1$. After some calculation we get to $E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})]] = \frac{B(a, b+\frac{\alpha-1}{m+1})}{\lambda^{\alpha-1}B(a,b)}$, finally, it can be obtained

$$H_\alpha(X_r) - H^\alpha_{U_r} = -\frac{1}{\alpha-1} log \frac{B(a, b+\frac{\alpha-1}{m+1})}{\lambda^{\alpha-1}B(a,b)}.$$

Theorem 5.3 For any random variable X, with density function $f(x)$, the standard uniform distribution has the minimum Renyi entropy of GOS.

Proof: Using Lemma (5.1) we have

$$H_\alpha(X_r) - H^\alpha_{U_r} = -\frac{1}{\alpha-1} log E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})]].$$

According to $0 \le f(x) \le 1$, and $\alpha > 1$ readily it can be shown $f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})] \le 1$, so, $-\frac{1}{\alpha-1} log E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})]] \ge 0$. Also, let $0 < \alpha < 1$ so $-\frac{1}{\alpha-1} > 0$, and $f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})] \ge 1$, using these inequalities we get to

$$-\frac{1}{\alpha-1} log E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z)^{\frac{1}{m+1}})]] \ge 0,$$

which completes the proof.

Definition 5.4 If $X \leq_{st} Y$ and φ be a non-decreasing (non-increasing) function, then $E(\varphi(X)) \leq (\geq) E(\varphi(Y))$. see Shaked and Shantikumar (1994).

Applying lemma (5.1) the Renyi entropy of $(r+1)th$ GOS is expressed as

$$H^{\alpha}_{X_{r+1}} = H^{\alpha}_{U_{r+1}} - \frac{1}{\alpha-1} log E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z_{r+1})^{\frac{1}{m+1}})]],$$

we define

$$\Delta^{\alpha}_{n}(r) = H^{\alpha}_{X_{r+1}} - H^{\alpha}_{X_r}$$

$$= \frac{1}{\alpha-1}\{\alpha \, log \frac{A_r}{A_{r+1}} + log L(\alpha,m,k) + log \frac{E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z_r)^{\frac{1}{m+1}})]]}{E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z_{r+1})^{\frac{1}{m+1}})]]}\}$$

wherein, $L(\alpha,m,k) = \dfrac{B(\alpha r+1-\alpha,b)}{B(\alpha r+1,b-\alpha)}$.

Theorem 5.5 Considering the assumptions of lemma 5.1, if $f(x)$ be a non-decreasing function in x, then $\Delta^{\alpha}_{n}(r) \leq 0$

Proof

Firstly, consider $\alpha > 1$. It can be shown that $\dfrac{A_r}{A_{r+1}} = \dfrac{r(m+1)}{\gamma_{r+1}}$, so $A_r \leq A_{r+1}$ for any $r \leq \dfrac{\gamma_1}{2(m+1)}$, also let $c = \alpha r + 1$ then $L(\alpha,m,k) = \dfrac{B(c-\alpha,b)}{B(c,b-\alpha)}$. Consider $G(w) = \dfrac{\Gamma(w-\alpha)}{\Gamma(w)}$, it is clear that $0 < G(w) < 1$, so we have $log\, G(x) < 0$, and using this fact that digamma function is a non-decreasing function it can be shown that $G(w)$ is a non-increasing function in w. In addition, it can be derived that $L(\alpha,m,k) = \dfrac{G(c)}{G(b)}$ thus, if $c > b$, that is, $r > \dfrac{\gamma_1}{2(m+1)} - \dfrac{1}{2}(\dfrac{1}{\alpha} + \dfrac{1}{m+1} - 1)$, then we get $log L(\alpha,m,k) < 0$. Afterwards, using assumption of the theorem, it is clear that for $\alpha > 1$ and by definition (5.4), it can be obtained

$$log \frac{E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z_r)^{\frac{1}{m+1}})]]}{E_{g(z)}[f^{\alpha-1}[F^{-1}(1-(1-Z_{r+1})^{\frac{1}{m+1}})]]} < 0,$$

so for any $\alpha > 0$, $\Delta^{\alpha}_{n}(r) \leq 0$.

Proof for the case $0 < \alpha < 1$ is similar to the previous case and is omitted.

Theorem 5.6 The Renyi information between distribution of GOS and data distribution is distribution free and is obtained as

$$D^{\alpha}_{n}(f_r(x) : f(x)) = -H^{\alpha}_{U_r}(x)$$

Proof: The result is implied By relation (7) and transformation $z = 1 - \overline{F}^{m+1}$.

Conclusion

In this paper, we explored informational properties of GOS, such as $Shannon$ and Renyi entropy, $Kullback-Leibler$, mutual and Renyi information. On the other hand, it showed that K-L, mutual and Renyi information between distribution of the GOS and data distribution is distribution free.

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